

Simple physical applications of a groupoid structure

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Abstract

Motivated by Quantum Mechanics considerations, we expose some cross product constructions on a groupoid structure. Furthermore, critical remarks are made on some basic formal aspects of the Hopf algebra structure.

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1. Introduction

In [11] was recovered the notion of *E-groupoid* [*EBB-groupoid*], whose groupoid algebra led to the so-called *E-groupoid algebra* [*EBB-groupoid algebra*]. Following a suggestion of A. Connes (see [6, § I.1]), some elementary algebraic structures of Matrix Quantum Mechanics can arise from certain representations of such groupoids and related group algebras. For instance, the map $(i, j) \rightarrow \nu_{ij}$ of [11, § 5], provides the kinematical time evolution of the observables \mathfrak{q} and \mathfrak{p} , given by the Hermitian matrices [11, § 5, (\boxtimes)], that, by means of the Kuhn-Thomas relation, satisfies the celebrated Heisenberg canonical commutation relations (see [23, § 14.4] and [3, § 2.3]), whereas the HBJ EBB-groupoid algebra constructed in [11, § 6], is the algebra of physical observables according to W. Heisenberg.

The structure of groupoid has many considerable applications both in pure and applied mathematics (see [4, 25]): here, let us consider the following, particular example, drew from Quantum Field Theory (QFT).

- *A toy model of QFT.* In Renormalization of Quantum Field Theory (see [26] and references therein), a Feynman graph Γ may be analytically represented as sum of iterated

divergent integrals defined as follows

$$\Gamma^1(t) = \int_t^\infty \frac{dp_1}{p_1^{1+\varepsilon}}, \quad \Gamma^n(t) = \int_t^\infty \frac{dp_1}{p_1^{1+\varepsilon}} \int_{p_1}^\infty \frac{dp_2}{p_2^{1+\varepsilon}} \dots \int_{p_{n-1}}^\infty \frac{dp_n}{p_n^{1+\varepsilon}} \quad n \geq 2$$

for $\varepsilon \in \mathbb{R}^+$; such integrals diverges logarithmically for $\varepsilon \rightarrow 0^+$.

It can be proved as these iterated integrals form a Hopf algebra of rooted trees (see [7, 26]), say \mathcal{H}_R , and called the *Connes-Kreimer Hopf algebra* (see [7]). The renormalization of these integrals requires a regularization, for instance through a linear multiplicative functional ϕ_a (bare Green function, defining a *Feynman rule*) defined on them, which represents a certain way of evaluation of the Feynman graphs, at the energy scale a , of the type

$$\phi_a \left(\prod_{i \in I} \Gamma^i(t) \right) = \prod_{i \in I} \Gamma^i(a),$$

where I denotes an arbitrary finite ordered subset of \mathbb{N} , and $\Gamma_a = \Gamma^0(a)$ are the normalized coupling constants at the energy scale (or renormalization point) a ; if Γ is a Feynman graph, then $\phi_a(\Gamma)$ is the corresponding regularized Feynman amplitude, according to the renormalization scheme parametrized by a .

So, every Feynman rule is a character $\phi_a : \mathcal{H}_R \rightarrow \mathbb{C}$ of the Hopf algebra \mathcal{H}_R , and their set is a (renormalization) group \mathcal{G}_R under the group law given by the usual convolution law $\hat{*}$ of \mathcal{H}_R . Thus, the coalgebra structure of \mathcal{H}_R endows \mathcal{G}_R with a well-defined group structure.

If S denotes the antipode of \mathcal{H}_R , then let us consider the following *deformed antipode* $S_a \doteq \phi_a \circ S$; in [26], it is considered a particular modification of the usual antipode axiom $S \hat{*} \text{id} = \text{id} \hat{*} S = \eta \circ \varepsilon$ (see [11, § 8]), precisely

$$\varepsilon_{a,b} = S_a \hat{*} \text{id}_b \doteq (\phi_a \circ S) \hat{*} \phi_b = m \circ (S_a \otimes \phi_b) \circ \Delta \quad (\text{renormalized Green functions}).$$

Hence, in [26, § 5], it is proved to be true the following pair groupoid law $\varepsilon_{a,b} \hat{*} \varepsilon_{b,c} = \varepsilon_{a,c}$, deduced from the Hopf algebra properties of \mathcal{H}_R . Moreover, if we consider the renormalized quantities $\varepsilon_{a,b}(\Gamma^n(t)) = \Gamma_{a,b}^n$, then we have

$$\Gamma_{a,b}^1 = \int_b^a \frac{dp}{p^{1+\varepsilon}}, \quad \Gamma_{a,b}^2 = \int_b^a \frac{dp_1}{p_1^{1+\varepsilon}} \int_{p_1}^a \frac{dp_2}{p_2^{1+\varepsilon}}, \quad \dots,$$

with every $\Gamma_{a,b}^n$ finite for $\varepsilon \rightarrow 0^+$, and zero for $a = b$. $\varepsilon_{a,b}$ is said to be a *renormalized character* of \mathcal{H}_R at the energy scales a, b . The correspondence (renormalization schemes) $\phi_b, \phi_b \rightarrow \varepsilon_{a,b}$ is what renormalization typically achieves.

Finally, from the relation $\varepsilon_{a,b} \hat{*} \varepsilon_{b,c} = \varepsilon_{a,c}$ and the coproduct rule of \mathcal{H}_R , it is possible to obtain the following relation

$$\Gamma_{a,c}^i = \Gamma_{a,b}^i + \Gamma_{b,c}^i + \sum_{j=1}^{i-1} \Gamma_{a,b}^j \Gamma_{b,c}^{i-j} \quad i \geq 2,$$

that is a generalization of the so-called *Chen's Lemma* (see [5, 13]); this relation describes what happens if we change the renormalization point.

If $\Gamma \in \mathcal{H}_R$ is a Feynman graph, then we have the following asymptotic expansion $\Gamma = \Gamma^0 + \Gamma^1 + \Gamma^2 + \Gamma^3 + \dots$; in general, such a series may be divergent, and, in this case, it can be renormalized to an finite, but undetermined, value. We have that $\varepsilon_{a,b}(\Gamma) = \Gamma_{a,b}$

is the result of the regularization of Γ at the energy scales a, b , and, respect to the scale change $\phi_a \rightarrow \phi_b$ (which allows us to renormalizes in a non-trivial manner), we have the following rule for the shift of the normalized coupling constants $\Gamma_b = \Gamma_a + \sum_{i \in \mathbb{N}} \Gamma_{a,b}^i$, in dependence of the running coupling constants $\Gamma_{a,b}^i$, $i \geq 1$.

In short, the comparison among different renormalization schemes (via the variation of the renormalization point) is regulated by the fundamental pair groupoid law $\varepsilon_{a,b} \hat{*} \varepsilon_{b,c} = \varepsilon_{a,c}$. Furthermore, we point out as this groupoid combination law, connected with a variation of the renormalization points, leads us to further formal properties of renormalization¹, as, for instance, the cohomological ones or the Callan-Symanzik type equations.

In [11, § 5], we have defined a specific EBB-groupoid, called the *Heisenberg-Born-Jordan EBB-groupoid* (or *HBJ EBB-groupoid*), whose group algebra, said *HBJ EBB-groupoid algebra*, may be endowed with a (albeit trivial) Hopf algebra structure, obtaining the so-called *HBJ EBB-Hopf algebra* (or *HBJ EBBH-algebra*), that is a first, possible example of generalization of the structure of Hopf algebra: indeed, it is a particular weak Hopf algebra, or quantum groupoid (see [19, § 2.5], [20, § 2.1.4] and [24, § 2.2]), in the finite-dimensional case.

We remember that group algebras were basic examples of Hopf algebras², so that groupoid algebras may be considered as basic examples of a class of structures generalizing the ordinary Hopf algebra structure; this class contains the so-called weak Hopf algebras, the Lu's and Xu's Hopf algebroids, and so on (see [1, 2, 14, 27]).

In this paper, starting from the E-groupoid structure exposed in [11], we want to introduce another, possible generalization of the ordinary Hopf algebra structure, following the notions of commutative Hopf algebroid (see [22]) and of quantum semigroup (see [9, § 1, p. 800]).

Moreover, at the end of [11, § 8], it has been mentioned both the triviality of the Hopf algebra structure there introduced (on the HBJ EBB-algebra), and some non-trivial duality questions related to the (possible) not finite generation of the HBJ EBB-algebra.

At the § 5. of the present paper³, we'll try to settle these questions by means of some fundamental works of S. Majid.

¹Moreover, it should be interesting to go into the question related to possible, further roles that the groupoid structures may play in Renormalization. See, also, the conclusions of § 7 of the present paper.

²See, for instance, the formalization of the quantum mechanics motivations adduced by V. G. Drinfeld in [9, § 1].

³That must be considered as a necessary continuation of [1].

Indeed, in [16] and [17], Majid has constructed non-trivial examples of non-commutative and non-cocommutative Hopf algebras (hence, non-trivial examples of quantum groups), via his notion of bicrossproduct. This type of structures involves group algebras and their duals; furthermore, these structures has an interesting physical meaning, since they are an algebraic representation of some quantum mechanics problems (see also [18, Chap. 6]).

Finally, we'll recall some other cross product constructions, among which the group Weyl algebra and the (Drinfeld) quantum double, that provides further, non-trivial examples of a quantum group having a really physical meaning.

Moreover, if we consider such structures applied to the EBJ EBBH-algebra (of [11, § 8]), then it is possible to get structures that represents an algebraic formalization of some possible quantum mechanics problems on a groupoid, in such a way to obtain new examples of elementary structures of a Quantum Mechanics on groupoids.

2. The Notion of E-semigroupoid

For the notions of E-groupoid and EBB-groupoid, with relative notations, we refer to [11, § 1].

An *E-semigroupoid* is an algebraic system of the type $(G, G^{(0)}, G^{(1)}, r, s, i, \star)$, where $G, G^{(0)}, G^{(1)}$ are non-void sets such that $G^{(0)}, G^{(1)} \subseteq G$, $r, s : G \rightarrow G^{(0)}$, $i : G^{(1)} \rightarrow G^{(1)}$ and $G^{(2)} = \{(g_1, g_2) \in G \times G; s(g_1) = r(g_2)\}$, satisfying the following conditions⁴:

- ₁ $s(g_1 \star g_2) = s(g_2), r(g_1 \star g_2) = r(g_1), \quad \forall (g_1, g_2) \in G^{(2)}$;
- ₂ $s(g) = r(g) = g, \quad \forall g \in G^{(0)}$;
- ₃ $g \star \alpha(s(g)) = \alpha(r(g)) \star g = g, \quad \forall g \in G$;
- ₄ $(g_1 \star g_2) \star g_3 = g_1 \star (g_2 \star g_3), \quad \forall g_1, g_2, g_3 \in G$;
- ₅ $\forall g \in G^{(1)}, \exists g^{-1} \in G^{(1)} : g \star g^{-1} = \alpha(r(g)), g^{-1} \star g = \alpha(s(g))$,

being $\alpha : G^{(0)} \hookrightarrow G$ the immersion of $G^{(0)}$ into G , and $i : g \rightarrow g^{-1}$. The maps r, s are called, respectively, *range* and *source*, G is the *support*, $G^{(0)}$ is the *set of units*, and $G^{(1)}$ is the *set of inverses*, of the given E-semigroupoid.

⁴Whenever the relative \star -products are well-defined.

For simplicity, we write $r(g), s(g)$ instead of $\alpha(r(g)), \alpha(s(g))$.

We obtain an E-groupoid when $G^{(1)} = G$ (see [11, § 1]), whereas we obtain a monoid when $G^{(0)} = \{e\}$. Moreover, if an E-semigroupoid also verify the condition of [11, § 1, •₆], then we have an *EBB-semigroupoid*.

3. The Notion of Linear \mathbb{K} -algebroid

A linear algebra (over a commutative scalar field \mathbb{K}) is an algebraic system of the type $(V_{\mathbb{K}}, +, \cdot, m, \eta)$, where $(V_{\mathbb{K}}, +, \cdot)$ is a \mathbb{K} -linear space and $(V, +, m)$ is a unital ring, satisfying certain compatibility conditions; in particular, (V, m) is a unital semigroup (that is, a monoid).

Following, in part, [21] (where it is introduced the notion of vector groupoid), if it is given an E-semigroupoid $(G, G^{(0)}, G^{(1)}, r, s, i, \star)$ such that

- i) $G_{\mathbb{K}} = (G, +, \cdot)$ is a \mathbb{K} -linear space, and $G^{(0)}, G^{(1)}$ are its linear subspaces;
- ii) r, s and i , are linear maps;
- iii) $g_1 \star (\lambda g_2 + \mu g_3 - s(g_1)) = \lambda(g_1 \star g_2) + \mu(g_1 \star g_3) - g_1$, $(\lambda g_1 + \mu g_2 - r(g_3)) \star g_3 = \lambda(g_1 \star g_3) + \mu(g_2 \star g_3) - g_3$, for every $g_1, g_2, g_3 \in G$ and $\lambda, \mu \in \mathbb{K}$ for which there exists the relative \star -products,

then we say that $(G_{\mathbb{K}}, G^{(0)}, G^{(1)}, r, s, i, \star)$ is a *linear \mathbb{K} -algebroid*.

4. The Notion of E-Hopf Algebroid

Let $\mathfrak{G}_{\mathbb{K}} = (G_{\mathbb{K}}, G^{(0)}, G^{(1)}, r, s, i, \star)$ be a linear \mathbb{K} -algebroid. If we set

$$G^{(2)} = \{g_1 \otimes_{\star} g_2 \in G \times G; s(g_1) = r(g_2)\} \doteq G \otimes_{\star} G,$$

$$m_{\star}(g_1 \otimes_{\star} g_2) = g_1 \star g_2,$$

then, more specifically, with $(\mathfrak{G}_{\mathbb{K}}, m_{\star}, \{\eta_r^{(e)}\}_{e \in G^{(0)}}, \{\eta_s^{(e)}\}_{e \in G^{(0)}})$, we'll denote such a linear \mathbb{K} -algebroid where, for each $e \in G^{(0)}$, we put $\eta_r^{(e)}, \eta_s^{(e)} : \mathbb{K} \rightarrow G$ in such a way that $\eta_r^{(e)}(k) = \{e\}$ and $\eta_s^{(e)}(k) = \{e\}$, $\forall k \in \mathbb{K}$. Hence, the unitary and associativity properties •₃ and •₄, of the given linear \mathbb{K} -algebroid, are as follows⁵

$$1. m_{\star} \circ (\eta_r^{(e)} \otimes_{\star} \text{id}) = m_{\star} \circ (\text{id} \otimes_{\star} \eta_s^{(e')}), \quad \forall e, e' \in G^{(0)},$$

⁵Henceforth, every partial \star -operation (as \otimes_{\star} , and so on) that we consider, it is assumed to be defined.

$$2. m_\star \circ (\eta_r^{(e)} \otimes_\star m_\star) = m_\star \circ (m_\star \otimes_\star \eta_s^{(e')}), \quad \forall e, e' \in G^{(0)},$$

where id is the identity of G .

Let us introduce, now, a cosemigroupoid structure as follows.

We define a partial comultiplication by a map $\Delta_\star : G \rightarrow G \otimes_\star G$, in such a way that, when the following condition holds:

$$3. (\text{id} \otimes_\star \Delta_\star) \circ \Delta_\star = (\Delta_\star \otimes_\star \text{id}) \circ \Delta_\star,$$

then we say that $(\mathfrak{G}_\mathbb{K}, \Delta_\star)$ is a *cosemigroupoid*.

If we require to subsist suitable homomorphism conditions for the maps $m_\star, \Delta_\star, \{\eta_r^{(e)}\}_{e \in G^{(0)}}, \{\eta_s^{(e)}\}_{e \in G^{(0)}}$, then we may to establish a certain *quantum semigroupoid* structure (in analogy to the quantum semigroup structure - see [9, § 1, p. 800]) on $(\mathfrak{G}_\mathbb{K}, m_\star, \{\eta_r^{(e)}\}_{e \in G^{(0)}}, \{\eta_s^{(e)}\}_{e \in G^{(0)}})$.

Following, in part, the notion of commutative Hopf algebroid given in [22, Appendix 1], if we define certain counits by maps $\varepsilon_r^{(e)}, \varepsilon_s^{(e)} : G \rightarrow \mathbb{K}$ for each $e \in G^{(0)}$, then we may to require that further counit properties holds, chosen among the following

$$4_1. (\text{id} \otimes_\star \varepsilon_r^{(e)}) \circ \Delta_\star = (\varepsilon_r^{(e')} \otimes_\star \text{id}) \circ \Delta_\star = \text{id}, \quad \forall e, e' \in G^{(0)},$$

$$4_2. (\text{id} \otimes_\star \varepsilon_s^{(e)}) \circ \Delta_\star = (\varepsilon_s^{(e')} \otimes_\star \text{id}) \circ \Delta_\star = \text{id}, \quad \forall e, e' \in G^{(0)},$$

$$4_3. (\text{id} \otimes_\star \varepsilon_r^{(e)}) \circ \Delta_\star = (\varepsilon_s^{(e')} \otimes_\star \text{id}) \circ \Delta_\star = \text{id}, \quad \forall e, e' \in G^{(0)},$$

$$4_4. (\text{id} \otimes_\star \varepsilon_s^{(e)}) \circ \Delta_\star = (\varepsilon_r^{(e')} \otimes_\star \text{id}) \circ \Delta_\star = \text{id}, \quad \forall e, e' \in G^{(0)},$$

with a set of compatibility conditions chosen among the following (or a suitable combination of them)

$$5_1. \varepsilon_r^{(e)} \circ \eta_r^{(e')} = \varepsilon_r^{(e)} \circ \eta_r^{(e)} = \text{id}, \quad \eta_r^{(e)} \circ \varepsilon_r^{(e')} = \eta_r^{(e)} \circ \varepsilon_r^{(e')} = \text{id}_{G^{(0)}}, \quad \forall e, e' \in G^{(0)},$$

$$5_2. \varepsilon_s^{(e)} \circ \eta_s^{(e')} = \varepsilon_s^{(e)} \circ \eta_s^{(e)} = \text{id}, \quad \eta_s^{(e)} \circ \varepsilon_s^{(e')} = \eta_s^{(e)} \circ \varepsilon_s^{(e')} = \text{id}_{G^{(0)}}, \quad \forall e, e' \in G^{(0)},$$

$$5_3. \varepsilon_r^{(e)} \circ \eta_s^{(e')} = \varepsilon_r^{(e)} \circ \eta_s^{(e)} = \text{id}, \quad \eta_r^{(e)} \circ \varepsilon_s^{(e')} = \eta_r^{(e)} \circ \varepsilon_s^{(e')} = \text{id}_{G^{(0)}}, \quad \forall e, e' \in G^{(0)},$$

$$5_4. \varepsilon_s^{(e)} \circ \eta_r^{(e')} = \varepsilon_s^{(e)} \circ \eta_r^{(e)} = \text{id}, \quad \eta_s^{(e)} \circ \varepsilon_r^{(e')} = \eta_s^{(e)} \circ \varepsilon_r^{(e')} = \text{id}_{G^{(0)}}, \quad \forall e, e' \in G^{(0)};$$

in such a case, we may define a suitable *linear \mathbb{K} -coalgebroid* structure of the type $(\mathfrak{G}_\mathbb{K}, \Delta_\star, \{\varepsilon_r^{(e)}\}_{e \in G^{(0)}}, \{\varepsilon_s^{(e)}\}_{e \in G^{(0)}})$, whence a *linear \mathbb{K} -coalgebroid* structure of the type $(\mathfrak{G}_\mathbb{K}, m_\star, \Delta_\star, \{\eta_r^{(e)}\}_{e \in G^{(0)}}, \{\eta_s^{(e)}\}_{e \in G^{(0)}}, \{\varepsilon_r^{(e)}\}_{e \in G^{(0)}}, \{\varepsilon_s^{(e)}\}_{e \in G^{(0)}})$.

If, when it is possible, we impose certain \mathbb{K} -algebroid homomorphism conditions for the maps $\Delta_\star, \{\varepsilon_r^{(e)}\}_{e \in G^{(0)}}, \{\varepsilon_s^{(e)}\}_{e \in G^{(0)}}$, and/or certain \mathbb{K} -coalgebroid homomorphism conditions for the maps $m_\star, \{\eta_r^{(e)}\}_{e \in G^{(0)}}, \{\eta_s^{(e)}\}_{e \in G^{(0)}}$, then we may to establish a certain *linear \mathbb{K} -bialgebroid* structure on $\mathfrak{B}_{\mathfrak{G}_{\mathbb{K}}}$, having posed $\mathfrak{B}_{\mathfrak{G}_{\mathbb{K}}} = (\mathfrak{G}_{\mathbb{K}}, m_\star, \Delta_\star, \{\eta_r^{(e)}\}_{e \in G^{(0)}}, \{\eta_s^{(e)}\}_{e \in G^{(0)}}, \{\varepsilon_r^{(e)}\}_{e \in G^{(0)}}, \{\varepsilon_s^{(e)}\}_{e \in G^{(0)}})$.

Finally, if, for each $f, g \in \text{End}(G_{\mathbb{K}})$ such that $f \otimes_\star g$ there exists, we put

$$G \xrightarrow{\Delta_\star} G \otimes_\star G \xrightarrow{f \otimes_\star g} G \otimes_\star G \xrightarrow{m_\star} G,$$

then it is possible to consider the following (partial) convolution product

$$f \tilde{\star}_\star g \doteq m_\star \circ (f \otimes_\star g) \circ \Delta_\star \in \text{End}(G_{\mathbb{K}}).$$

Thus, an element $a \in \text{End}(G_{\mathbb{K}})$ may to be said an *antipode* of a \mathbb{K} -bialgebroid structure $\mathfrak{B}_{\mathfrak{G}_{\mathbb{K}}}$, when there exists $a \tilde{\star}_\star \text{id}, \text{id} \tilde{\star}_\star a$, and

$$a \tilde{\star}_\star \text{id} = \text{id} \tilde{\star}_\star a = 2^{th} \text{ condition of } 5_i,$$

if $\mathfrak{B}_{\mathfrak{G}_{\mathbb{K}}}$ has the property 5_i , $i = 1, 2, 3, 4$.

A \mathbb{K} -bialgebroid structure with, at least, one antipode, is said to have an *E-Hopf algebroid* structure. If $G^{(0)} = \{e\}$, then we obtain an ordinary Hopf algebra structure.

5. The Majid's quantum gravity model

S. Majid, in [16] and [17], have introduced a particular noncommutative and noncocommutative bicrossproduct Hopf algebra that should be viewed as a toy model of a physical system in which both quantum effects (the noncommutativity) and gravitational curvature effects (the noncocommutativity) are unified. The Majid construction, being a noncommutative noncocommutative Hopf algebra, may be viewed as a non-trivial example of quantum group having an important physical meaning. We'll apply this model to the EBJ EBB-groupoid algebra (eventually equipped with a riemannian structure).

Following [15, Chap. III, § 1], an E-groupoid $(G, G^{(0)}, r, s, \star)$ is said to be a *differentiable E-groupoid* (or a *E-groupoid manifold*) if $G, G^{(0)}$ are differentiable manifolds, the maps $r, s : G \rightarrow G^{(0)}$ are surjective submersions, the inclusion $\alpha : G^{(0)} \hookrightarrow G$ is smooth, and the partial multiplication $\star : G^{(2)} \rightarrow G$ is smooth (if we understand $G^{(2)}$ as submanifold of $G \times G$).

A locally trivial⁶ differentiable E-groupoid is said to be a *Lie E-groupoid*.

Following [10], a differentiable E-groupoid $(G, G^{(0)}, r, s, \star)$ is said to be a *Riemannian E-groupoid* if there exists a metric g over G and a metric g_0 over $G^{(0)}$ in such a way that the inversion map $i : G \rightarrow G$ is an isometry, and r, s are Riemannian submersions of (G, g_0) onto $(G^{(0)}, g_0)$.

Let $\mathcal{A}_{HBJ}(= \mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I)) \cong \mathcal{A}_{\mathbb{K}}(\mathcal{G}_{Br}(I)))$ be the HBJ EBB-algebra of [11, § 6]: as seen there, it represents the algebra of physical observables according to the Matrix Quantum Mechanics.

The question of which metric to choose, for instance over \mathcal{A}_{HBJ} , or over $\mathcal{G}_{HBJ}(= \mathcal{G}_{HBJ}(\mathcal{F}_I) \cong \mathcal{G}_{Br}(I))$ (for this last groupoid isomorphism, see [11, § 5]), is not trivial and not a priori dictated (see, [12, II] for a discussion of a similar question related to a tentative of metrization of the symplectic phase-space manifold of a dynamical system, in order that be possible to define a classical Brownian motion on it).

Let us introduce a Majid's toy model of quantum mechanics combined with gravity, following [16, 17, 18].

S. Majid ([16]) follows the abstract quantization formulation of I. Segal, whereby any abstract \mathbb{C}^* -algebra can be considered as the algebra of observables of a quantum system, and the positive linear functionals on it as the states.

He, first, considers a pure algebraic formulation of the classical mechanics of geodesic motion on a Riemannian spacetime manifold, precisely on a homogeneous spacetime, following the well-known *Mackey's quantization procedure* on homogeneous spacetimes (see [18, Chap. 6], and references therein).

The basis of the Majid's physical picture lies in a new interpretation of the semidirect product algebra as quantum mechanics on homogeneous spacetime (according to [8]). Namely, he considers the semidirect product $\mathbb{K}[G_1] \ltimes_{\alpha} \mathbb{K}(G_2)$ where G_1 is a finite group that acts, through α , on a set G_2 ; here, $\mathbb{K}[G_1]$ denotes the group algebra over G_1 , whereas $\mathbb{K}(G_2)$ denotes the algebra of \mathbb{K} -valued functions on G_2 .

[17, section 1.1] motivates the search for self-dual algebraic structures in general, and Hopf algebras in particular, so that it is natural to search the self-dual structure of $\mathbb{K}[G_1] \ltimes_{\alpha} \mathbb{K}(G_2)$, as follows.

⁶For the notion of local triviality of a topological groupoid, see [15, Chap. II, § 2].

To this end, we assume that G_2 is also a group that acts back by an action β on G_1 as a set; so, one can equivalently view that β induces a coaction of $\mathbb{K}[G_2]$ on $\mathbb{K}(G_1)$, and defines the corresponding semidirect coproduct coalgebra which we denote $\mathbb{K}[G_1]^\beta \rtimes \mathbb{K}(G_2)$. Such a bicrossproduct structure will be denoted $\mathbb{K}[G_1]^\beta \bowtie_\alpha \mathbb{K}(G_2)$.

Majid's model fit together the semidirect product by α with the semidirect coproduct by β , to form a Hopf algebra; in such a way, we'll have certain (compatibility) constraints on (α, β) that gives a *bicrossproduct* Hopf algebra structure to $\mathbb{K}[G_1] \otimes \mathbb{K}(G_2)$, that it is of self-dual type; this structure is non-commutative [non-cocommutative] when α [β] is non-trivial.

Majid's model of quantum gravity starts from the physical meaning of a bicrossproduct structure relative to the case $\mathbb{K} = \mathbb{C}$, $G_1 = G_2 = \mathbb{R}$ and $\alpha_{\text{left } u}(s) = \hbar u + s$, with \hbar a dimensionful parameter (Planck's constant), achieved as a particular case of the classical self-dual $*$ -Hopf algebra of observables (according to I. Segal) $\mathbb{C}^*(G_1) \otimes \mathbb{C}(G_2)$, where G_1, G_2 has a some group structure and $\mathbb{C}^*(G_1)$ is the convolution \mathbb{C}^* -algebra on G_1 . Further, with a suitable compatibility conditions (see [16, (9)] or [18, (6.15)]) for (α, β) – that can be viewed as certain (Einstein) "second-order gravitational field equations" for α (that induces metric properties on G_2), with back-reaction β playing the role of an auxiliary physical field – we obtain a bicrossproduct Hopf algebra $\mathbb{C}^*(G_1)^\beta \bowtie_\alpha \mathbb{C}(G_2)$, with the following self-duality $(\mathbb{C}^*(G_1)^\beta \bowtie_\alpha \mathbb{C}(G_2))^* \cong \mathbb{C}^*(G_2)^\alpha \bowtie_\beta \mathbb{C}(G_1)$.

Moreover (see [16]), in the Lie group setting, the non-commutativity of G_2 (whence, the non-cocommutativity of the coalgebra structure) means that the intrinsic torsion-free connection on G_2 , has curvature (cogravity), that is to say, the non-cocommutativity plays the role of a Riemannian curvature on G_2 (in the sense of non-commutative geometry).

We, now, consider a simple quantization problem (see [17, § 1.1.2]). Let $G_1 = G_2$ be a group and α the left action; hence, the algebra $\mathcal{W}(G) \doteq \mathbb{K}^*[G] \rtimes_{\text{left}} \mathbb{K}(G)$ will be called the *group Weyl algebra* of G , and it represents the algebraic quantization of a particle moving on G by translations.

Finally, if we want to apply these considerations to the case $G_1 = G_2 = \mathcal{G}_{HBJ}$, then we must consider both the finite-dimensional and the infinite-dimensional case, in such a way that be possible to determine the dual (or the restricted dual) of $\mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ})$.

Taking into account the physical meaning of \mathcal{G}_{HBJ} , it follows that it is possible to consider the above mentioned algebraic structures (with their physical meaning) in relation to the case study⁷ $G_1 = G_2 = \mathcal{G}_{HBJ}$, with consequent physical interpretation (where possible), in such a way to get non-trivial examples of quantum groupoids⁸ having a possible quantic meaning.

6. A particular Weyl Algebra (and other structures)

The cross and bicross (or double cross) constructions provides the basic algebraic structures on which to build up non-trivial examples of quantum groups, even in the infinite-dimensional case.

In this paragraph, we expose some examples of such constructions.

Let $\mathcal{F}_{HBJ} \doteq \mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}(\mathcal{F}_I))$ be the linear \mathbb{K} -algebra of \mathbb{K} -valued functions defined on \mathcal{G}_{HBJ} .

Let (G, \cdot) be a finite group. If $\mathcal{F}_{\mathbb{K}}(G)$ is the linear \mathbb{K} -algebra of \mathbb{K} -valued functions on G , then, since $\mathcal{F}_{\mathbb{K}}(G) \otimes \mathcal{F}_{\mathbb{K}}(G) \cong \mathcal{F}_{\mathbb{K}}(G \times G)$, it follows that such an algebra can be endowed with a natural structure of Hopf algebra by the following data

1. coproduct $\Delta : \mathcal{F}_{\mathbb{K}}(G) \rightarrow \mathcal{F}_{\mathbb{K}}(G \times G)$ given by $\Delta(f)(g_1, g_2) = f(g_1 \cdot g_2)$ for all $g_1, g_2 \in G$;
2. counit $\varepsilon : \mathcal{F}_{\mathbb{K}}(G) \rightarrow \mathbb{K}$, with $\varepsilon(f) = 1$ for each $f \in G$;
3. antipode $S : \mathcal{F}_{\mathbb{K}}(G) \rightarrow \mathcal{F}_{\mathbb{K}}(G)$, defined by $S(f)(g) = f(g^{-1})$ for all $g \in G$.

If we consider a finite groupoid instead of a finite group, then 3. even subsists, but 1. and 2. are no longer valid because of the groupoid structure.

If $\mathcal{G} = (G, G^{(0)}, r, s, \star)$ is a finite groupoid, then, following [24, § 2.2], the most natural coalgebra structure on $\mathcal{F}_{\mathbb{K}}(\mathcal{G})$, is given by the following data

- 1'. coproduct $\Delta(g_1, g_2) = f(g_1 \star g_2)$ if $(g_1, g_2) \in G^{(2)}$, and $= 0$ otherwise;
- 2'. counit $\varepsilon(f) = \sum_{e \in G^{(0)}} f(e)$.

⁷Such a groupoid may be, eventually, endowed with a further topological and/or metric (as the Riemannian one) structure.

⁸If we consider a non-commutative non-cocommutative Hopf algebra as a model of quantum group.

In such a way, via 1', 2' and 3, it is possible to consider a Hopf algebra structure on $\mathcal{F}_{\mathbb{K}}(\mathcal{G})$.

In the finite-dimensional case, we have $\mathcal{A}_{\mathbb{K}}^*(\mathcal{G}) \cong \mathcal{F}_{\mathbb{K}}(\mathcal{G})$ (see [18, Example 1.5.4]) as regard the dual Hopf algebra $\mathcal{A}_{\mathbb{K}}^*(\mathcal{G})$ of $\mathcal{A}_{\mathbb{K}}(\mathcal{G})$, whereas, in the infinite-dimensional case, we have the restricted dual $\mathcal{A}_{\mathbb{K}}^*(\mathcal{G}) \cong \mathcal{F}_{\mathbb{K}}^{(o)}(\mathcal{G}) \subseteq \mathcal{F}_{\mathbb{K}}(\mathcal{G})$; hence, in our case $\mathcal{G} = \mathcal{G}_{HBJ}$, following [18, Chap. 6] it is possible to consider a non-degenerate dual pairing, say $\langle \cdot, \cdot \rangle$, between $\mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ})$ and $\mathcal{F}_{\mathbb{K}}^{(o)}(\mathcal{G}_{HBJ})$ (in the finite-dimensional case, it is $\mathcal{F}_{\mathbb{K}}^{(o)}(\mathcal{G}_{HBJ}) = \mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ})$). Hence, if we define the action

$$\alpha : (b, a) \rightarrow b \triangleright a = \langle b, a_{(1)} \rangle a_{(2)} \quad \forall a \in \mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ}), \quad \forall b \in \mathcal{F}_{\mathbb{K}}^{(o)}(\mathcal{G}_{HBJ}),$$

then it is possible to define the left cross product algebra

$$\mathcal{H}(\mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ})) \doteq \mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ}) \ltimes_{\alpha} \mathcal{F}_{\mathbb{K}}^{(o)}(\mathcal{G}_{HBJ}),$$

called the *Heisenberg double* of $\mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ})$.

If V is an A -module algebra, with A a Hopf algebra, then let $V \ltimes A$ be the corresponding left cross product, and $(v \otimes a) \triangleright w = v(a \triangleright w)$ the corresponding Schrödinger representation (see [18, § 1.6]) of V on itself. If V has a Hopf algebra structure and $V^{(o)}$ is its restricted dual via the pairing $\langle \cdot, \cdot \rangle$, then (see [18, Chap. 6]) the action σ given by $\phi \triangleright v = v_{(1)} \langle \phi, v_{(2)} \rangle$ for all $v \in V, \phi \in V^{(o)}$, make V into a $V^{(o)}$ -module algebra and $V \otimes V^{(o)}$ into an algebra with product given by

$$(v \otimes \phi)(w \otimes \psi) = vw_{(1)} \otimes \langle w_{(2)}, \phi_{(1)} \rangle \phi_{(2)} \psi,$$

so that let $\mathcal{W}(V) \doteq V \ltimes_{\sigma} V^{(o)}$ be the corresponding left cross product algebra. Hence, it is possible to prove (see [18, Chap. 6]) that the related Schrödinger representation (see [18, § 6.1, p. 222]) give rise to an algebra isomorphism $\chi : V \ltimes_{\sigma} V^{(o)} \rightarrow \text{Lin}(V)$ (= algebra of \mathbb{K} -endomorphisms of V), given by $\chi(v \otimes \psi)w = vw_{(1)} \langle \phi, w_{(2)} \rangle$; $\mathcal{W}(V) \doteq V \ltimes_{\sigma} V^{(o)}$ is said to be the (*restricted*) *group Weyl algebra* of the Hopf algebra V .

This last construction is an algebraic generalization of the usual Weyl algebra of Quantum Mechanics on a group (see [17, § 1.1.2]), whose finite-dimensional prototype is as follows.

Let G be a finite group, and let's consider the strict dual pair given by the \mathbb{K} -valued functions on G , say $\mathbb{K}(G)$, and the free algebra on G , say $\mathbb{K}G$. Hence, the right action of G on itself given by $\psi_u(s) = su$, establishes a left

cross product algebra structure, say $\mathbb{K}(G) \ltimes \mathbb{K}G$, on $\mathbb{K}(G) \otimes \mathbb{K}G$; such an action induces (see [18, Chap. 6]), also, a left regular representation of G into $\mathbb{K}G$, so that we can consider the related Schrödinger representation generated by it and by the action of $\mathbb{K}G$ on itself by pointwise product. Thus, if $V = \mathbb{K}(G)$, with $\mathbb{K}(G)$ endowed with the usual Hopf algebra structure, then we have that the Weyl algebra $\mathbb{K}(G) \ltimes \mathbb{K}G$ (with $V^{(o)} = \mathbb{K}(G)^* = \mathbb{K}G$ since G is finite) is isomorphic to $\text{Lin}(\mathbb{K}(G))$ via the Schrödinger representation. As already said in the previous paragraph, such a Weyl algebra formalizes the algebraic quantization of a particle moving on G by translations.

If we apply what has been said above to \mathcal{G}_{HBJ} in the finite-dimensional case (that is, when $\text{card } I < \infty$ that correspond to a finite number of energy levels – see [11, § 6]), since $V^{(o)} = \mathcal{F}_{\mathbb{K}}^{(o)}(\mathcal{G}_{HBJ}) = \mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ})$, then we have that

$$\mathcal{W}(\mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ})) = \mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}) \ltimes \mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ})$$

represents the algebraic quantization of a particle moving on the groupoid \mathcal{G}_{HBJ} by translations⁹, remembering the quantic meaning (according to I. Segal) of $\mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ})$ (as set of states) and $\mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ})$ (as set of observables). Instead, in the infinite-dimensional case, we have that $\mathcal{F}_{\mathbb{K}}^{(o)}(\mathcal{G}_{HBJ})$ is isomorphic to a sub-Hopf algebra of $\mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ})$ (this is the HBJ EBBH-algebra – see [11, § 8]), so that

$$\mathcal{W}(\mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ})) = \mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}) \ltimes \mathcal{F}_{\mathbb{K}}^{(o)}(\mathcal{G}_{HBJ}) \subseteq \mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}) \ltimes \mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ}).$$

Finally, the right adjoint action (see [18, § 1.6]) of \mathcal{G}_{HBJ} on itself given by $\psi_g(h) = g^{-1} \star h \star g$ if there exists, and $= 0$ otherwise, make $\mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ})$ into an $\mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ})$ -module algebra. In the finite-dimensional case, we have $\mathcal{A}_{\mathbb{K}}^*(\mathcal{G}_{HBJ}) \cong \mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ})$, so that

$$\mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ}) = \mathcal{A}_{\mathbb{K}}^{**}(\mathcal{G}_{HBJ}) \cong \mathcal{F}_{\mathbb{K}}^*(\mathcal{G}_{HBJ}),$$

whence $\mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ})$ is a $\mathcal{F}_{\mathbb{K}}^*(\mathcal{G}_{HBJ})$ -module algebra too. Therefore, we may consider the following left cross product algebra

$$\mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}) \ltimes \mathcal{F}_{\mathbb{K}}^*(\mathcal{G}_{HBJ}) \cong \mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}) \ltimes \mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ})$$

that the tensor product coalgebra makes into a Hopf algebra, called the (*Drinfeld*) *quantum double* of \mathcal{G}_{HBJ} , and denoted $\mathcal{D}(\mathcal{G}_{HBJ})$; even in the finite-dimensional case, it represent the algebraic quantization of a particle constrained to move on conjugacy classes of \mathcal{G}_{HBJ} (quantization on homogeneous

⁹From here, we may speak of a Quantum Mechanics on a groupoid.

space over a groupoid).

Besides, it has been proved, for a finite group G , that this (Drinfeld) quantum double $\mathcal{D}(\mathcal{G}_{HBJ})$, has a quasitriangular structure (see [18, Chap. 6]) given by

$$\begin{aligned}(\delta_s \otimes u)(\delta_t \otimes v) &= \delta_{u^{-1}su, t} \delta_t \otimes uv, & \Delta(\delta_s \otimes u) &= \sum_{ab=s} \delta_a \otimes u \delta_b \otimes u, \\ \varepsilon(\delta_s \otimes u) &= \delta_{s, e}, & S(\delta_s \otimes u) &= \delta_{u^{-1}s^{-1}u} \otimes u^{-1}, \\ R &= \sum_{u \in G} \delta_u \otimes e \otimes 1 \otimes u,\end{aligned}$$

where we have identifies the dual of $\mathbb{K}G$ with $\mathbb{K}(G)$ via the idempotents $p_g, g \in G$ such that $p_g p_h = \delta_{g, h} p_g$ (see [19, § 2.5] and [18, § 1.5.4]). Such a quantum double represents the algebra of quantum observables of a certain physical system with symmetry group G .

Hence, even in the finite-dimensional case¹⁰, we may consider, with suitable modifications, an analogous quasitriangular structure on \mathcal{G}_{HBJ} , obtaining the (Drinfeld) quantum double $\mathcal{D}(\mathcal{G}_{HBJ})$ on \mathcal{G}_{HBJ} ; thus, if we consider a quasitriangular Hopf algebra as a model of quantum group, the (Drinfeld) quantum double $\mathcal{D}(\mathcal{G}_{HBJ})$ provides a non-trivial example of quantum group having a (possible) quantic meaning (related to a quantum mechanics on a groupoid).

7. Conclusions

From what has been said above, and in [11], it rises a possible role played by the groupoid structures in Quantum Mechanics and Quantum Field Theory.

For instance, such a groupoid structures¹¹ might takes place a prominent rule in Renormalization, as well as in the Majid's model of quantum gravity. Indeed, a central problem in quantum gravity concerns its nonrenormalizability due to the existence of UV divergences; in turn, the UV divergences arise from the assumption that the classical configurations being summed over are defined on a continuum.

So, the discreteness given by groupoid structures may turn out to be of some usefulness in such a (renormalization) QFT problems.

¹⁰The infinite-dimensional case is not so immediate.

¹¹Eventually equipped with further, more specific structures, as the topological or metric ones.

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